Mathematics Extended Essay

Title: The History of Complex Numbers

Research Question: What problems led to the discovery of complex numbers and their properties?

Word Count: 3,597

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Introduction

Before algebraic notation was invented, the Babylonians were able to be the first to solve the quadratic by completing the square. They, like the majority of other mathematicians in the 17th century, visualized the quadratic as an area problem. When solving for the area or a side length, having negative results would not be logical; thus, for several centuries, mathematicians were averse to the notion of having negative number solutions. It is no surprise then, that the stigma was even worse when it came to seeing negative numbers inside of a square root.

Although there are claims that imaginary numbers can trace its origins to when Heron of Alexandria solved for the volume of a frustum or when Hindu mathematician, Brahmagupta recognized negative numbers solutions to the quadratic, it was the need to solve cubic equations that led to their discovery and subsequent development. Similar to how there were different forms of the quadratic arranged to only have positive coefficients, mathematician Omar Khayyam (11th century) listed 19 different versions of the cubic. Khayyam solved for some forms of the cubic by finding the intersection between hyperbolas and circles, which future mathematicians would implement in the development of complex numbers as well. It was only when the need to find a general solution to the cubic that led mathematicians to take a second look at what would become complex numbers.

In this essay, I will describe and discuss the problems and discoveries that various mathematicians encountered overtime to arrive and affirm the existence of complex numbers and their properties. I will also work through some of the problems that these mathematicians did to demonstrate how they obtained their discoveries relating to complex numbers.

The Need to Solve Cubic Equations

The birth of imaginary numbers began in Renaissance Italy with many mathematicians attempting to solve the problem that Luca Pacioli dubbed as "impossible". During the 16th century, and for many years after, the majority of mathematicians would believe that there is no solution to the cubic. When Scipione del Ferro discovered the method on how to solve a special case of the general form of a cubic, specifically known as the depressed cubic. A depressed cubic is a cubic where the quadratic term is absent.

With the general form of a cubic being: $ax^3 + bx^2 + cx + d = 0$, it can be reduced to: $x^3 + cx + d = 0$, through a change in variable, where $x' = x + \frac{b}{3}$. This substitution was mentioned in two anonymous Florentine manuscripts near the end of the 14th century.

$$a(x' - \frac{b}{3})^3 + b(x' - \frac{b}{3})^2 + c\left(x' - \frac{b}{3}\right) + d$$

$$ax'^3 - ax'^2b + \frac{ax'^{b^2}}{3} - \frac{ab^3}{27} + x'^2b - \frac{2x'^{b^2}}{3} + \frac{b^2}{9} + cx' - \frac{cb}{3} + d$$

If a is a constant and equals 1 then,

$$x'^3 - x'^2b + \frac{x'b^2}{3} - \frac{b^3}{27} + x'^2b - \frac{2x'^{b^2}}{3} + \frac{b^3}{9} + cx' - \frac{cb}{3} + d$$

$$x'^3 - \frac{x'b^2}{3} - \frac{b^3}{27} + \frac{b^3}{9} + cx' - \frac{cb}{3} + d$$

$$x'^3 + x'\left(c - \frac{b^2}{3}\right) + \left(\frac{2b^3}{27} - \frac{bc}{3} + d\right)$$

The x'^2b terms cancel out, so this condenses to the general form of $x^3 + cx + d$.

This leads to three cases of the depressed cubic (where c and d are positive, as mathematicians during this time did not accept negative numbers):

1.) $x^3 + cx = d$
2.) $x^3 = cx + d$
3.) $x^3 + d = cx$

Starting with the first case of the depressed cubic (since it is unclear whether Ferro was able to solve the second and third case to the depressed cubic or not):

$$x^3 + cx = d$$

Ferro substituted $u + v$ for x, which results in:

$$(u + v)^3 + (u + v)c = d$$

$$u^3 + 3u^2v + 3uv^2 + v^3 + uc + vc = d \text{ (when fully expanded)}$$

This expanded form, however, would not be useful for Ferro, which is why he continued to factor and simplify it to include an additional constraint:

$$u^3 + v^3 + 3uv(u + v) + (u + v)c = d$$

$$u^3 + v^3 + (u + v)(3uv + c) = d$$

The additional constraint allowed Ferro to add more variables into the equation with his substitution, which now allowed him to rewrite the equation into two separate equations:

1.) $u^3 + v^3 = d$
2.) $3uv + c = 0$

Using these two equations, solving for *v* in the second equation would make it possible to substitute it back into the first equation, which is what Ferro did to arrive to a general solution to a depressed cubic.

$$v = -\frac{c}{3u}$$

$$u^3 + (-\frac{c}{3u})^3 = d$$

$$u^3 - \frac{c^3}{27u^3} = d$$

Multiplying by u^3 to eliminate the variable in the denominator.

$$u^6 - \frac{c^3}{27} = du^3$$

$$p = u^3, q = d, r = -\frac{c^3}{27}$$

$$p^2 - dp + r = 0$$

Using the quadratic formula:

$$p = \frac{q \pm \sqrt{q^2 - 4r}}{2} = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} - r}$$

$$u^3 = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}$$

$$u = \sqrt[3]{\frac{d}{2} \pm \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}}$$

Ferro only accepted the positive root in this case due to the continuing hostility against negative numbers.

With u, v can be found to arrive to Ferro's general solution ($u^3 + v^3 = d$):

$$v = \sqrt[3]{d - (\frac{d}{2} + \sqrt{\frac{q^2}{4} + \frac{c^3}{27}})} = \sqrt[3]{\frac{d}{2} - \sqrt{\frac{q^2}{4} + \frac{c^3}{27}}}$$

Since $x = u + v$, this gives Ferro's general solution to the depressed cubic, which gives one real and two complex solutions.

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}}$$

After Ferro's death in 1526, mathematician, Niccolo Fontana, also known as "Tartagalia" openly proclaimed that he could solve the depressed cubic, as well as the cubic in the form of $ax^3 + bx^2 = d$. Fontana approached the cubic by extending the idea of completing the square into 3-dimensions, which can be thought of as a volume problem, where a cube with sides x plus a volume of the x term is equivalent to the constant (for the depressed form). However, in order to "complete the square", Tartagalia needed to add on to the cube by extending the cube by a length of the x term.

Visual:

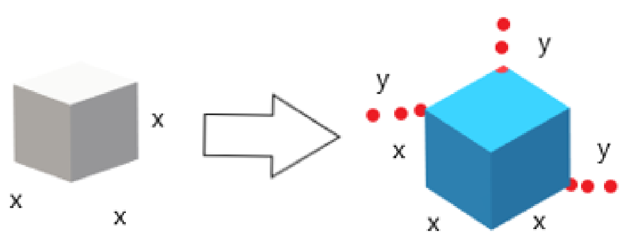

Figure 1 Diagram I made on Paint

This forms a new, larger cube with a total length of “z”, where $z = x + y$. When the new cube is broken up, there are 3 prisms with dimensions x by x by y, another 3 prisms (x by y by y), and a cube with sides y. Tartagalia groups these 6 rectangular prisms with a new dimension of x by z (z = x + y), and 3y. The volume formula obtained from this: (V=3yzx) can then be substituted into the x term, where 3yzx = coefficient of the x term. Finally, to complete the “cube”, Tartagalia simply added y^3 to both sides of the equation.

This revelation intrigued another fellow mathematician, Girolamo Cardano, who rediscovered Ferro’s surviving works, as well as persuaded Tartagalia to reveal his secrets for solving the cubic. With this knowledge, as well as independently working on his own method, Cardano published in his *Ars Magna* (1545) (giving inspirational credit to Ferro and Tartagalia), his formula for solving depressed cubics, which also extended to all cubics, as well. His solution to the depressed cubic is very similar to that of Ferro’s, but his approach was different. (cont.)

He started with because the general form of the cubic where he made numbered a to represent the different terms: $ax^3 + a_1x^2 + a_2x + a_3 = 0$, he substituted in for *x*:

$$x = y - \frac{a_1}{3}$$

$$a\left(y - \frac{a_1}{3}\right) + a_1\left(y - \frac{a_1}{3}\right) + a_2\left(y - \frac{a_1}{3}\right) + a_3$$

After expanding and simplifying this equation:

$$ay^3 - ay^2a_1 + \frac{ay(a_1)^2}{3} - \frac{a(a_1)^3}{27} + a_1y^2 - \frac{2(a_1)^2y}{3} + \frac{(a_1)^3}{9} + a_2y - \frac{a_2a_1}{3} + a_3$$

$$In\ Cardano's\ case, a\ is\ also\ a\ constant\ equal\ to\ 1.$$

$$y^3 - a_1y^2 + \frac{(a_1)^2y}{3} - \frac{(a_1)^3}{27} + a_1y^2 - \frac{2(a_1)^2y}{3} + \frac{(a_1)^3}{9} + a_2y - \frac{a_2a_1}{3} + a_3$$

$$y^3 - \frac{(a_1)^2y}{3} + \frac{2(a_1)^3}{27} + a_2y - \frac{a_1a_2}{3} + a_3$$

$$y^3 - y\left(a_2 - \frac{(a_1)^2}{3}\right) + (\frac{2(a_1)^3}{27} - \frac{a_1a_2}{3} + a^3)$$

$$n = a_2 - \frac{(a_1)^2}{3}, m = \frac{2(a_1)^3}{27} - \frac{a_1a_2}{3} + a^3$$

$$\therefore y^3 + my + n = 0$$

After arriving to these values, Cardano substituted these values back into Ferro's formula for a depressed cubic: (cont.)

$$x = \sqrt[3]{\frac{m}{2} + \sqrt{\frac{m^2}{4} + \frac{n^3}{27}}} + \sqrt[3]{\frac{m}{2} - \sqrt{\frac{m^2}{4} + \frac{n^3}{27}}}$$

$$x = \sqrt[3]{\frac{(\frac{2(a_1)^3}{27} - \frac{a_1 a_2}{3} + a^3)}{2} + \sqrt{\frac{(\frac{2(a_1)^3}{27} - \frac{a_1 a_2}{3} + a^3)^2}{4} + \frac{(a_2 - \frac{(a_1)^2}{3})^3}{27}}}$$
$$+ \sqrt[3]{\frac{(\frac{2(a_1)^3}{27} - \frac{a_1 a_2}{3} + a^3)}{2} - \sqrt{\frac{(\frac{2(a_1)^3}{27} - \frac{a_1 a_2}{3} + a^3)^2}{4} + \frac{(a_2 - \frac{(a_1)^2}{3})^3}{27}}}$$

Despite being the first to formulate a general solution to the cubic equation, Cardano realized that if the secondary square root in the general solution was less than 0, it would result in the square root of a negative number. In his *Ars Magna*, he provides an example of this in his problem of "diving 10 into two parts whose product is 40" or $x^2 - 10x + 40 = 0$. Arriving to the solutions: $5 - \sqrt{-15}$ and $5 + \sqrt{-15}$, he stated that the complex roots seemed so "impossible" that they were "useless". However, what Cardano stumbled upon happens to be a complex conjugate pair and this would help with their continued development by Rafael Bombelli.

The Development of the Arithmetic Properties of Complex Numbers

Initially skeptical of complex numbers, calling them “more sophistic than true”, it was Rafael Bombelli who introduced the notation for the imaginary number and contributed to the development of their arithmetic properties. Having studied Girolamo Cardano’s work in cubics, Bombelli knew that complex numbers were essential to solve not only cubics, but also quartics equations. Despite, not receiving a university education, Bombelli wanted to write an algebra book that would resolutely explain mathematical concepts to prevent the plethora of ongoing arguments between mathematicians during the 16^{th} century. In *L'Algebra* (1572), Bombelli tackles the animosity surrounding negative numbers and the idea of the “imaginary number” at the time by outlining the rules for their arithmetic calculations. For negative numbers, he listed the results for their arithmetic operations (e.g. “minus times minus makes plus”), which would help demonstrate that imaginary numbers also follow some of the same rules. In his book, the specific properties that he discusses for complex numbers are: the addition and subtraction of complex numbers and the multiplication of complex numbers. The properties of the conjugate of a complex number are also touched upon in his book when he explored Cardano’s solution to a cubic further.

1.) The Properties of Addition and Subtraction of Complex Numbers

When outlining the rules for adding and subtracting complex numbers, Bombelli made sure to note that the real parts can only be operated with the real parts and the same follows for the imaginary parts, as well. Furthermore, since Bombelli demonstrated that arithmetic operations can be performed with complex numbers, this means that they hold true for the commutative and associative property for addition and subtraction, as well as holds true for the identity laws of addition and the law of the additive inverse.

Let:

$$z = a + bi$$
$$z' = a' + bi'$$
$$z'' = a'' + bi''$$

Then:

$$z + z' = z' + z \ (Commutative)$$
$$(z + z') + z'' = z + (z' + z'') \ (Associative)$$
$$z = z + 0 = 0 + z \ (Identity\ Law)$$
$$0 = -z + z = z + \ -z \ \ (Law\ of\ Additive\ Inverse)$$

2.) The Properties of Multiplication of Complex Numbers

Similar to addition, only corresponding components (either real or imaginary) can be operated with one another. The laws of arithmetic and algebra also apply to complex numbers being multiplied.

$$z \cdot z' = z' \cdot z \ (Commutative)$$
$$(z \cdot z') \cdot z'' = z \cdot (z' \cdot z'') \ (Associative)$$
$$z = z \cdot 1 = 1 \cdot z \ (Identity\ Law)$$
$$1 = z \cdot \frac{1}{z} = \frac{1}{z} \cdot z \ (Law\ of\ Multiplicative\ Inverse)$$

3.) The Properties of the Conjugate of Complex Numbers (The Irreducible Case)

In *L'Algebra*, Bombelli further explores Cardano's formula for cubics by showing that complex roots can produce real solutions, despite the formula resulting in the square root of a negative number. He starts by giving the equation: $x^3 = 15x + 4 = x^3 - 15x = 4\ (in\ depressed\ cubic\ form)$.

Using Cardano's formula(refer back to pg.9), where m=4 and n=-15

$$x = \sqrt[3]{\frac{m}{2} + \sqrt{\frac{m^2}{4} + \frac{n^3}{27}}} + \sqrt[3]{\frac{m}{2} - \sqrt{\frac{m^2}{4} + \frac{n^3}{27}}}$$

$$x = \sqrt[3]{\frac{4}{2} + \sqrt{\frac{(4)^2}{4} + \frac{(-15)^3}{27}}} + \sqrt[3]{\frac{4}{2} - \sqrt{\frac{(4)^2}{4} + \frac{(-15)^3}{27}}}$$

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Cardano labeled this sum of two complex conjugates as the "irreducible root". However, Bombelli saw how this case can lead to three real roots. Polynomial division reveals that there are actually three real roots to this cubic equation:

x=4 is a solution to the equation, as can be seen through substitution $((4)^3 - 15(4) = 4)$. From here, polynomial division can be used to show that there are three real solutions to the equation:

$$x^2 + 4x + 1$$

$$x - 4 \overline{\smash{)}\, x^3 - 15x - 4}$$

$$-(x^3 - 4x^2)$$

$$4x^2 - 15x - 4$$

$$-(4x^2 - 16x)$$

$$x - 4$$

$$-(x - 4)$$

$$0$$

Using the quadratic equation to solve $x^2 + 4x + 1 = 0$:

$$x = \frac{-4 \pm \sqrt{12}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

To show that there are three real solutions to the equation, Bombelli conceived the idea to reduce the solution that Cardano's formula produced to the form: $a + b\sqrt{-1}$, as he had already come up with the imaginary notation. This meant that:

$$\sqrt[3]{2 + \sqrt{-121}} = a + b\sqrt{-1}$$

$$\sqrt[3]{2 - \sqrt{-121}} = a - b\sqrt{-1}$$

For this to be true, Bombelli made the following assumptions:

$$b\sqrt{-1} + b'\sqrt{-1} = (b + b')\sqrt{-1}$$

$$b\sqrt{-1} \cdot b'\sqrt{-1} = (-1)(b \cdot b')$$

$$a + bi = c + di;\ a = c, b = d$$

With this, the first two statements become:

$$2 + \sqrt{-121} = (a + b\sqrt{-1})^3$$

$$2 - \sqrt{-121} = \ (a - b\sqrt{-1})^3$$

From here, binomial expansion can be used, knowing that $(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$.

So, let $u = a\ and\ v = b\sqrt{-1}$.

$$a^3 + 3a^2\left(b\sqrt{-1}\right) + 3a(b\sqrt{-1})^2 + (b\sqrt{-1})^3$$

$$a^3 + 3a^2\left(b\sqrt{-1}\right) - 3ab^2 - b^3\sqrt{-1}$$

$$a(a^2 - 3b^2) + b(3a^2 - b^2)\sqrt{-1})$$

Now that the expansion is simplified, its components can be set equal to the "complex number".

Simplifying $2 + \sqrt{-121} = 2 + 11i$.

$$\therefore a(a^2 - 3b^2) = 2, b(3a^2 - b^2) = 11$$

Since 2 and 11 are primes, they both share a factor of 1, which means that a=2 and b=1.

$$\therefore 2^2 - 3(1)^2 = 1,$$

$$3(2)^2 - (1)^2 = 11$$

Hence,

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + \sqrt{-1}$$

$$\sqrt[3]{2 - \sqrt{-121}} = 2 - \sqrt{-1}$$

Referring back to the original solution obtained from Cardano's formula, the reduced form of it can be substituted back in:

$$x = 2 + \sqrt{-1} + 2 - \sqrt{-1}$$

Since, the $\sqrt{-1}$ cancels out, x=4, which was one of the real roots obtained previously.

Ultimately, through this problem, Bombelli not only demonstrated that complex roots can produce real solutions, but he also introduced the complex conjugate, which can lead to many unexpected occurrences.

A surprising occurrence that the conjugate of a complex number leads to is the generation of Pythagorean triples through the division of their conjugate pairs. Part of the reason for this is because of the connection between the formula for finding the modulus of a complex number and the Pythagorean theorem, where both formulas give magnitude to a certain distance.

Given $\frac{a+bi}{a-bi}$

$= \frac{a+bi}{a-bi} \times \frac{a+bi}{a+bi}$, where a, b ⊂ R

$= \frac{a^2+2abi-b^2}{a^2+b^2}$

$= \frac{(a^2-b^2)+(2ab)i}{a^2+b^2}$

Using the Pythagorean Theorem:

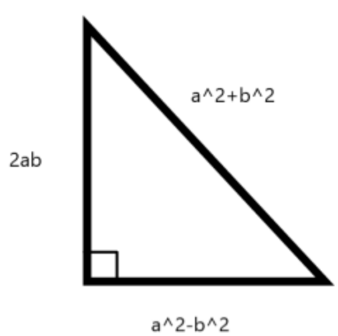

Figure 2 Diagram I created on Paint App

$$(a^2 - b^2)^2 + (2ab)^2$$

$$= a^4 - 2a^2b^2 + b^4 + 4a^2b^2$$

$$= a^4 + 2a^2b^2 + b^4$$

$$= (a^2 + b^2)^2$$

4.) The Properties of Division of Complex Numbers

The development of the complex conjugate makes it possible to divide complex numbers, as the quotient of two complex numbers is obtained by multiplying the numerator and denominator by the conjugate if the denominator.

5.) Powers of Complex Numbers

Euler's formula allows for complex numbers to be expressed as exponents:

$$e^{ix} = cosx + isinx = e^{\pi i} + 1 = 0$$

His formula helped to involve the use of polar coordinates in the complex plane. To arrive to this formula, Euler first noticed that $sinx$ consists of odd powers and $cosx$ consists of even powers, while e^x consists of both; hence, if $sinx + cosx = e^x$:

$$sinx = 0 + x \quad - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$+$

$$\cos x = 1 \quad - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$1 + x - - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!}$$

The sign changes (positive to negative) that occur as a result of this correlate to the powers of imaginary numbers ($i^2 = -1, i^3 = -i, etc$) and because of this Euler replaced the e^x terms for e^{ix}:

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{(ix)^3}{3!} + \frac{x^4}{4!} + \frac{(ix)^5}{5!} - \frac{x^6}{6!} + \cdots$$

The remaining $i's$ in the odd powers can be fixed by changing $\sin x$ to $isinx$.

It is important to note that Abraham De Moivre was one the first mathematicians to implement complex numbers in trigonometry with his formula: $(cosx + isinx)^n$ (however, he did not write his formula explicitly). His formula helps to solve for powers of complex numbers.

6.) Polar Property of Complex Numbers

Although Euler visualized complex numbers as rectangular coordinates, he never established the polar form for them; he only defined the complex exponential using his formula. The polar form of a complex number can be found by taking its modulus and finding its argument using arctan.

The Need to Geometrically Represent Complex Numbers

While Cardano and Bombelli introduced complex numbers and their properties, René Descartes still felt that they lacked physical interpretation, which is why in his treatise, *La Géometrie* (1637), he made a connection between imaginary numbers and their "geometric impossibility". In his work, he frequently uses the negative square root, later coining the term, "imaginary" for them.

Descartes established that positive roots from quadratics were the only ones that had geometric significance, which makes sense because, as one does not think of area, for example, as negative. To demonstrate how complex number intertwine with "geometric impossibility", he began by solving the quadratic equation: $z^2 = az - b^2$ geometrically. He started with the construction of a line segment $\overline{AB}$ and extending that to $\overline{AC}$, with B as the center of the circle, and $\frac{1}{2}a = \overline{AB}\ and\ b = \overline{AC}$:

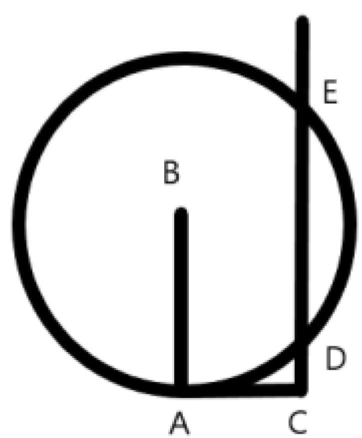

Figure 3 Descartes' construction I created on Paint App

Although Descartes did not include a proof of his work for this problem, mathematicians assumed that from here, he considered the ΔBDE:

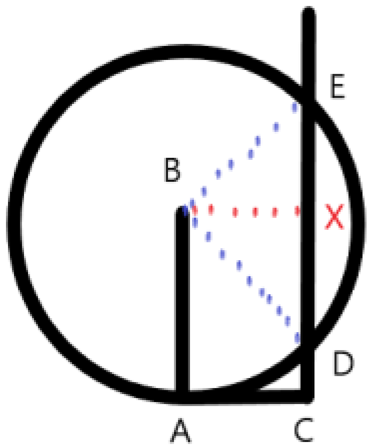

Figure 4 Diagram I created on Paint app

From here, it can be seen that $\overline{CD} = \frac{1}{2}a - \frac{1}{2}\overline{DE}$ and that $\overline{CE} = \overline{CD} + \overline{DE}$. So, $\overline{DE}$ can be found by using the Pythagorean Theorem:

$$(\overline{DX})^2 = \overline{(BD)}^2 - (\overline{BX})^2$$

$$(\overline{DX})^2 = (\frac{1}{2}a)^2 - (b^2)$$

$$\overline{DX} = \sqrt{\frac{1}{4}a^2 - b^2}$$

$$\overline{CD} = \overline{CX} - \overline{DX}$$

$$\overline{CD} = \frac{1}{2}a - \sqrt{\frac{1}{4}a^2 - b^2}$$

$$\overline{CE} = \overline{CX} + \overline{XE}, \overline{DX} = \overline{XE}$$

$$\overline{CE} = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 - b^2}$$

Based on his geometric construction of the quadratic, Descartes concluded that $\overline{CD}$ and $\overline{CE}$ are the two roots, as they touch the line passing through the circle. With this solution, he showed that complex roots were possible (unlike what he thought at the time), as if $a < b$, then, this leads to a square root of a negative number.

The Ambiguous Case

Despite what Descartes believed about complex numbers and their "geometric impossibility", John Wallis carried forth the progress of geometrically representing them. In his correspondence with another mathematician, John Collins, Wallis described a problem with a triangle with sides of 1 and 2 and a base of 4. Collins responded back to him saying that if he proceeded any further, he would encounter the square root of a negative number, which at the time was still associated with the "impossible". This made Wallis more inclined to dispel the aversion associated with negative numbers. He pictured negative numbers as the distance away from zero, which led to the idea of the number line, with positive numbers to the right of the line and negative numbers to the left. With a more solid foundation, he continued working on his ambiguous case problem.

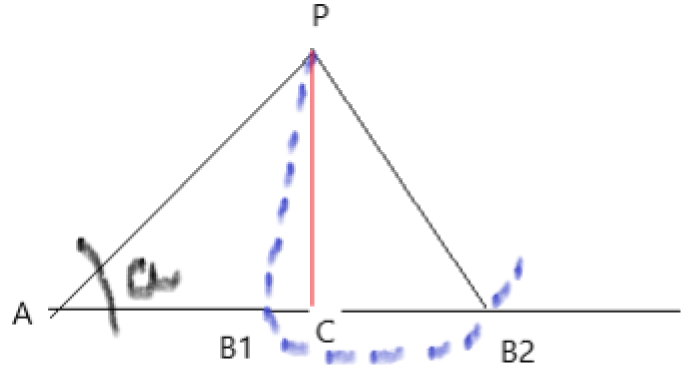

Figure 5 Diagram of Wallis's ambiguous case I created on Paint App

The diagram shown above can have two possible solutions and in this case they are the sides: AB1 and AB2.

Side AB1 and AB2 can be found by:

$$AB1 = AC - B1C$$

$$AC = \sqrt{(PA)^2 - (PC)^2}$$

$$B1C = \sqrt{(PB1)^2 - (PC)^2}$$

$$\therefore AB1 = (\sqrt{(PA)^2 - (PC)^2}) - (\sqrt{(PB1)^2 - (PC)^2}\,)$$

$$AB2 = AC + CB2$$

$$CB2 = \sqrt{(PB2)^2 - (PC)^2}$$

$$\therefore AB2 = (\sqrt{(PA)^2 - (PC)^2}) + (\sqrt{(PB2)^2 - (PC)^2})$$

Seeing that if PB1 and PB2 < PC would lead to the square root of a negative number, Wallis wanted to continue even under that condition, and so he extended points B1 and B2 away from AD, then the solutions would still be “possible”. He did that by moving points B1 and B2 up from AD and drew a circle with PC as the diameter, as shown:

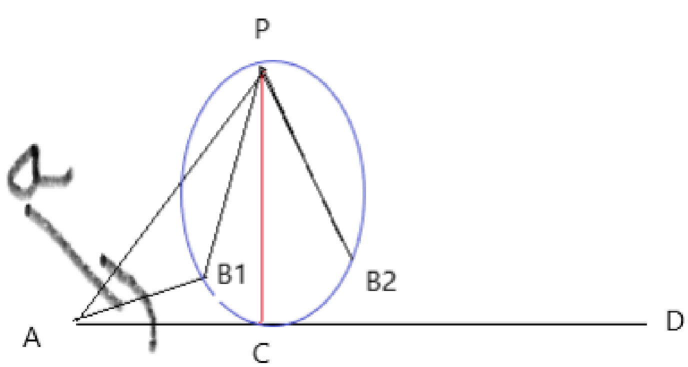

Figure 6 Diagram of Wallis's construction I created on Paint App

By moving points B1 and B2 up, Wallis demonstrated that imaginary numbers can be geometrically represented on the vertical axis of a plane, which would later appear in the Argand diagram. Although, he never stated this, his construction demonstrated a possible understanding of this idea that would be further developed by Caspar Wessel and Jean-Robert Argand in the later 18th century.

Wrapping Up Wallis's Case

Caspar Wessel is credited as the mathematician who finalized a way to geometrically represent complex numbers, as he continued what John Wallis did by thinking of complex numbers as a point $(a + bi)$ that could be represented by a vector from the origin to that point. In 1799, he published his paper, "On the Analytic Representation of Direction: An Attempt", where he uses vector methods to approach complex numbers, such as the parallelogram method. He also defined the multiplication of vectors by adding polar angles and multiplying the magnitudes in his paper. However, it was Sir Isaac Newton who first created the polar coordinate system and Leonhard Euler who laid the foundation for complex numbers to be expressed in polar form.

To begin wrapping up Wallis's case, Wessel began by showing vector addition via the parallelogram method. The parallelogram method works by placing two vectors on the same initial point and drawing a diagonal at where the lines meet (as shown): He also demonstrated this using the triangular method as well, where the heads and tails of the segments line up. From there, he established that the addition of complex numbers should follow the same set of rules as the vectors.

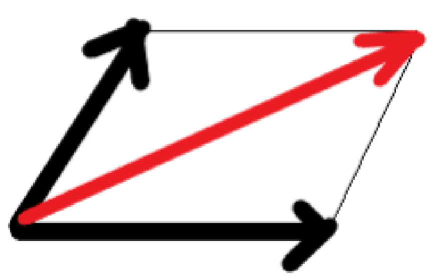

Figure 7 Parallelogram Method Diagram Example I made on Paint App

One of Wessel's more notable contributions to the development of complex numbers came with his work in multiplying vectors. His generalization of real numbers helped him to apply that to complex numbers, which led him to say that the length of the product of two complex numbers is equal to the product of the lengths. As for the direction of the vector, Wessel added the angle of one with the other.

For example: $(2 + i)(4 + i) = 7 + 6i$

The modulus of $(2 + i)$ is $\sqrt{5}$ and $(4 + i)$ is $\sqrt{17}$. The modulus of $(7 + 6i)$ is $\sqrt{85}$ and the product of $\sqrt{5} \cdot \sqrt{17}$ is $\sqrt{85}$. The argument of $(2 + i)$ is $\tan^{-1}\left(\frac{1}{2}\right) = 0.464$ and the argument of $(4 + i)$ is $\tan^{-1}\left(\frac{1}{4}\right) = 0.245$. The argument of $(7 + 6i)$ is $\tan^{-1}(\frac{6}{7}) = 0.709$, which is the sum of 0.464 and 0.245.

Wessel also built upon this by developing a more efficient way to solve complex numbers being raised to a power by finding the modulus of the complex number and raising that to n power, followed by finding the argument and multiply that by n. For example:

$$(2 + i)^{12}$$

Modulus: $\sqrt{5}$, Argument: $\tan^{-1}\left(\frac{1}{2}\right) = 0.464$.

$$(\sqrt{5})^{12} = 15625$$
$$(0.464)(12) = 5.568$$

Hence, Wessel's discovery of multiplying the moduli of the complex numbers and adding their arguments, which in turn is equivalent to the length and direction helped to establish geometric meaning to $\sqrt{-1}$.

However, what really solidified the representation of the imaginary number on the vertical axis of a plane was when Wessel took $\sqrt{-1}$ with modulus r and argument θ and multiplied that by itself, which equals: $-1, r^2, 2\theta$ and so: $-1 = \cos(180°)$, which means: $2\theta = 180°$ and $\theta = 90°$. This leads to the conclusion that the $\sqrt{-1}$ is a 90°.

Argand's Diagram

Similar to Wessel, Jean-Robert Argand also published a method for representing complex numbers geometrically in 1806. As it often happens in mathematics, both Argand and Wessel published their works around the same time, but independently of one another. However, today the complex plane is better known as Argand's diagram, rather than Wessel's.

Subsequent Developments with Complex Numbers

A few years after Argand's publication, another mathematician named Jacques Français continued to expand on the applications of the geometrical representation of complex numbers, as well as detail the basics of complex numbers in his paper, (Annales de Mathémathiques (1813)). Subsequently, a few decades later, Carl Friedrich Gauss would officially introduce the term, "complex number" and Augustin-Louis Cauchy would pioneer the field of complex analysis.

Conclusion

Ultimately, the problems that led to the discovery of complex numbers and their properties began with the need to solve cubic equations. The discovery of the general formula to solve cubic equations led to other implications concerning complex numbers, as it led to the appearance of the complex conjugate. As mathematicians became more accepting of negative numbers and the acknowledging the possibility of a complex solution, complex numbers started to get used more frequently in arithmetic operations. Later, the need arose to not only represent complex numbers algebraically, but also geometrically. This led to the development how imaginary numbers would be represented on the vertical axis on the complex plane. With more physical meaning given to complex numbers, they were able to be applied to operations with vectors and trigonometry. Today, complex numbers have moved past their initial reluctance since the 16th century, as they are commonly used in electrical engineering, quantum physics, and more. Overall, complex numbers allow mathematicians to solve for solutions that were once not obtainable before.

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